Gauge theory of disclinations on fluctuating elastic surfaces

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Abstract

A variant of a gauge theory is formulated to describe disclinations on Riemannian surfaces that may change both the Gaussian (intrinsic) and mean (extrinsic) curvatures, which implies that both internal strains and a location of the surface in R^3 may vary. Besides, originally distributed disclinations are taken into account. For the flat surface, an extended variant of the Edelen-Kadić gauge theory is obtained. Within the linear scheme our model recovers the von Karman equations for membranes, with a disclination-induced source being generated by gauge fields. For a single disclination on an arbitrary elastic surface a covariant generalization of the von Karman equations is derived.

1 Introduction

Elastic two-dimensional structures which are free to change their geometry (membranes, thin films, etc.) as well as deformable materials with spherical or tubular shapes (fullerenes, nanotubes) are of considerable current interest (see, e.g., [1, 2, 3] and the references therein). The properties of these crystalline structures are found to be essentially affected by their topology. It has been found that topological defects, first of all disclinations, play an important role in these objects. In particular, the Kosterlitz–Thouless disclination unbinding transition in hexatic membranes was shown to depend on shape fluctuations [4, 5] since a membrane with a single disclination can lower its energy by buckling. Fullerenes and nanotubes always contain at least 12 disclinations on their surfaces (i.e. 5-fold coordinated sites) as a consequence of the Euler theorem. The effect of shape fluctuations on the interaction of the disclinations on a spherical surface with genus zero was studied in [6].

Elastic models for membranes and shells are well known (see, e.g., [7, 8] and the references therein). The main problem is how to incorporate defects into the elastic theory of two-dimensional fluctuating surfaces. A possible way has been considered in [1] where defects were introduced *ad hoc* as source terms in the right-hand side of the von Karman equations. This approach is efficient in description of monolayers as well as membranes under condition that the bending rigidity which controls out-of-plane fluctuations is small. Similar approaches were developed in [4] where the Coulomb-like model was formulated. Disclinations were introduced there as the point-like charges on the curved surface.

The modern trends in theoretical description of topological defects in condensed matter include geometrical and gauge-theory methods (see, e.g., [11, 12, 13]). In the present paper we put forward a gauge theory that enables one to describe disclinations on $2D$ elastic surfaces that may change both their intrinsic and extrinsic geometry. Namely, both the internal strain and the location of the surface in R^3 may vary. By analogy with the Edelen–Kadić (EK) gauge model of dislocation and disclinations [14, 15], defects are incorporated via dynamical gauge fields. However, we formulate some principal statements which provide quite a new geometrical setting to handle a problem of describing defect-developing deformations of an elastic body. As is shown below, this allows to describe defects on arbitrarily fluctuating surfaces as well as dynamics of originally distributed defects. Notice that even in the $2D$ planar case our theory does not identically coincides with that obtained within the EK approach [19]. As a matter of fact, it includes the latter as a particular case and allows for a possibility to include objects with originally distributed disclinations.

A basic motivation for our investigation has been that of extending the EK gauge theory to include nontrivial geometry in the $2D$ case, e.g., that of membranes, spheres, etc. It should be emphasized that within the standard EK approach there is no room for such objects at all. Indeed, within that approach deformation is mathematically described as a diffeomorphic mapping of a defect-free *domain* $D_0 \in R^n$ which is usually called a reference configuration to another one $D \in R^n$ (current configuration) with noneucledian metric tensor g. By the very definition D_0 and D necessarily have the same dimension n, thereby ruling out the possibility to consider dynamics of, say, defects on $2D$-surfaces for they are not domains in R^3.

Any attempt at generalizing the EK theory to study disclinations on general manifolds should imply two steps. First, one has to appropriately reformulate a classical theory of elasticity and, second, to introduce in thus obtained new geometrical setting dynamical gauge fields. A straightforward generalization might be that of considering a diffeomorphic map $\chi :\, D_0 \in \Sigma \to D \in \Sigma$, where Σ stands for a Riemannian surface [1] (i.e., a $2D$ real Riemannian manifold) located in R^3. This map can (locally) be described by functions $\chi^a(x^b)\,(a, b = 1, 2)$, where x^a denotes a point in D_0 and χ^a corresponds to coordinates in D. A resulting Lagrangian to describe elastic properties of the media would follow as a function of the state vector $\chi^a(x)$, which would result in a kind of a field theory in the nontrivial geometrical background. To include defects an appropriate gauge field on a curved space Σ would be further required, an obvious complication in contrast with the EK approach.

The present approach has been proven to adequately incorporate dynamics of disclinations on varying elastic surfaces as well as to take into account originally distributed defects. The key observation that considerably simplifies and at the same time generalize the matter is that one should consider a surfaces Σ being embedded into a three-dimensional flat space R^3 instead of proceeding entirely in terms of its intrinsic geometry. As we shall see shortly, following this idea will lead us to a fairly plain and surprisingly complete theory of defect dynamics on curved surfaces.

The paper is structured as follows. In section 2 we formulate the gauge model of disclinations on $2D$ elastic surfaces. The action which includes elastic deformations, self-energy of disclinations, and the curvature energy is constructed in a self-consistent way. A complete set of equations of motion is presented in section 3. To illustrate the model, we consider three examples in section 4. First, we derive equations of motion for the planar case and show that they are distinct from those obtained within the EK approach. Second, we study a problem of the

[1]In particular, Σ may denote a *Riemann* surface, that is a $1D$ compact orientable complex manifold.

fluctuating surface by employing the linear approximation. In this case, the known equations for fluctuating membranes are recovered in a self-consistent way with a source formed by the gauge disclination fields. Finally, a concrete realization of the model for a single disclination on arbitrary elastic surface is presented. Section 5 is devoted to concluding comments.

2 The model

Before proceeding, a few comments on the limitations of the theory as well as on the conventions employed are to be made. First of all, we are solely concerned with the $2D$ case, although our formalism can easily be extended to any space dimensions. Our motivation is that the $2D$ case is both the most important in applications and at the same time the simplest one in notation.[2] Secondly, we take into consideration only the rotational symmetry of a system thereby making an attempt at describing disclinations and leaving aside dislocations and other defects. By turning to the full-fledged internal symmetry group instead of the orthogonal one the above restriction might be avoided, though it will evidently entail considerable conventional complications. Finally, static configurations are only considered, which does not seem, however, to be a deficiency in the following exposition, since including time evolution brings in no novel features compared to the EK theory.

An action that is assumed to properly describe dynamics of disclinations on a deformable elastic surface is taken in the form

$$S = S_{el} + S_{gauge} + S_{fl}, \tag{1}$$

where S_{el} describes the elastic properties of the media, S_{gauge} stands for the action of a gauge field that incorporates self-action for disclinations, and S_{fl} is the Helfrich-Canham action [9, 10] to describe the energy of a free fluctuating surface.

Let us start by discussing the first piece of the action. Let $x^a (a = 1, 2)$ be a set of local coordinates on a certain Riemannian surface Σ_0. (Indices $a, b, c, ... = 1, 2$ are tangent to Σ_0, whereas $i, j, k, ... = 1, 2, 3$ run over the basis of R^3). Under a deformation, Σ_0 is assumed to evolve into some other surface Σ. To describe this we find it convenient to introduce embeddings Σ_0, $\Sigma \to R^3$ that can be realized in terms of two R^3-valued functions $R^i_{(0)}(x^1, x^2)$ and $R^i(x^1, x^2)$, respectively. As the point (x^1, x^2) is varied vectors $\vec{R}_{(0)}$ and $\vec{R}$ sweep surfaces Σ_0 and Σ, respectively. This is nothing but a familiar two-parametric representation of surfaces in R^3, the point, however, being that

$$\vec{R}(x) := \phi^* \vec{R}_{(0)} = \vec{R}_{(0)}[\phi(x)], \tag{2}$$

where ϕ^* is a pullback of $\phi : \Sigma_0 \to \Sigma$. In what follows functions $R^i_{(0)}(x^1, x^2)$ are chosen to specify an initial configuration Σ_0, whereas dynamical variables $R^i(x^1, x^2)$ are taken to describe the deformation $\Sigma_0 \to \Sigma$. With these conventions at hand, a proper generalization of the elasticity theory turns out to be a straightforward matter.

Representations for the induced metrics follow immediately

$$\begin{aligned} g_{ab} &\equiv (g_{\Sigma_0})_{ab} = \partial_a \vec{R}_{(0)} \cdot \partial_b \vec{R}_{(0)}, \\ \tilde{g}_{ab} &\equiv (\phi^* g_\Sigma)_{ab} = (g_\Sigma)_{cd} \frac{\partial \phi^c}{\partial x^a} \cdot \frac{\partial \phi^d}{\partial x^b} = \frac{\partial \vec{R}}{\partial \phi^c} \cdot \frac{\partial \vec{R}}{\partial \phi^d} \frac{\partial \phi^c}{\partial x^a} \cdot \frac{\partial \phi^d}{\partial x^b} = \partial_a \vec{R} \cdot \partial_b \vec{R}, \end{aligned} \tag{3}$$

[2] $1D$ case is irrelevant for us here since there are no $1D$ objects with an intrinsic curvature

where the set $\{\phi^a\}$ stands for local coordinates on Σ. The strain tensor is determined to be [16, 17]

$$E_{ab} = \tilde{g}_{ab} - g_{ab}.$$

The elastic properties of the deformed surface are described by an action

$$S_{el} = -\frac{1}{8}\int_{\Sigma_0} dx^1 dx^2 \sqrt{g}\left\{\lambda(trE)^2 + 2\mu\, trE^2\right\}, \tag{4}$$

where $trE = g^{ab}E_{ab}$, $g = det||g_{ab}||$ and summation over repeated indices is assumed. Let us mention that we omit in (4) the terms of order E^3 and higher. In some special cases they can be considered as well (see, e.g., [15]).

To proceed, a properly defined gauge field to describe disclinations is to be introduced. To make this point transparent, let us step a bit aside and discuss the principle of a local gauge invariance from the geometric viewpoint. In this regard, a simple example may be of some help. Consider a scalar complex field $(\psi, \bar{\psi}): \; x \in R^n \to C$. Let a Lagrangian exhibit a global $U(1)$ symmetry: $L \to L$ under a transformation $\psi \to e^{i\alpha}\psi$, $\bar{\psi} \to e^{-i\alpha}\bar{\psi}$. In what way the local gauge fields can then be introduced? To this end, one considers ψ and $\bar{\psi}$ as a description not of a map $R^n \to C$ but of sections of a line C-bundle over R^n with the structure group $U(1)$. This is a trivial bundle which admits global sections. A connection on this bundle [3] is a familiar gauge field A_μ which can be regarded as a $U(1)$ valued one-form. The resulting theory is nothing but scalar electrodynamics.

In order to incorporate disclinations in the elasticity theory (4), one should as the above example suggests consider the R^3-vector bundle over Σ and the $R^3_{(0)}$-bundle over Σ_0 with the same structure groups $SO(3)$. The $so(3)$ valued one form $A^{(0)}_a(x)dx^a$ ($A^{(0)}_a = \vec{W}^{(0)}_a \cdot \vec{L}$, $\quad L^i \in so(3)$) serves as a connection one-form in the $R^3_{(0)}$-bundle space over Σ_0, with $\vec{W}^{(0)}_a$ being the gauge potentials. A connection on the R^3-bundle over Σ_0 is obtained by pulling back the connection of the R^3-bundle over Σ:

$$\vec{W}_a := \phi^* \vec{W}_a\mid_\Sigma = \partial_a\phi^b(\vec{W}_b\mid_\Sigma).$$

By replacing in (4) ordinary derivatives $\partial_a\vec{R}$ and $\partial_a\vec{R}_{(0)}$ by the covariant ones $\nabla_a\vec{R} = \partial_a\vec{R} + [\vec{W}_a, \vec{R}]$ and $\nabla_a\vec{R}_{(0)} = \partial_a\vec{R}_{(0)} + [\vec{W}^{(0)}_a, \vec{R}_{(0)}]$, respectively, one arrives at the desired locally $SO(3)$ invariant representation for the elasticity Lagrangian.

A few remarks are in order at this stage. First, we consider potentials $\vec{W}^{(0)}_a$ as given fixed functions to describe disclinations originally distributed on Σ_0. These potentials being involved in a full set of equations of motion, provide a possibility to keep track of the dynamics of these disclinations under deformation. Such a possibility is missing in the standard EK theory where from the very beginning only defect-free initial configurations are allowed.

Second, although essentially two-dimensional manifolds are considered, *three*-dimensional rotational group $SO(3)$ is involved, which seems to be quite natural in the framework of our approach. Enlarging the structure group to the semidirect product $SO(3) \triangleright T(3)$, where $T(3)$ stands for the group of translations in R^3, enables one to include into consideration both disclinations and dislocations.

Third, if we made an attempt to formulate an appropriate gauge theory in the scope of the direct generalization of the EK approach, we would encounter the following problem. With

[3]To be more precise, a connection one-form is defined on the associated principal bundle $P(R^n; U(1))$ and completely specifies the covariant derivative on a C-bundle over R^n.

the $\chi^a(x)$ being not the Σ-valued functions defined on Σ but local sections of a Σ-bundle over Σ, the derivatives $\partial_a \chi_b$ must be replaced by suitable covariant derivatives. What would be the "gauge group" in the definition of these covariant derivatives? In general this would be the infinite dimensional group Diff Σ of all diffeomorphisms of Σ which is a fibre of the bundle. For an arbitrary Σ it almost surely does not include rotations in contrast to the group Diff $R^3 = GL(3, R) \ni SO(3)$. To find a way out, one might be tempted to relate defects to a group of all isometries of (Σ, g) rather than to $SO(3)$. This group belongs to Diff Σ and is generated by operators $\hat{V}_a$ obeying

$$\mathcal{L}_{V_a} g = 0,$$

where $\mathcal{L}_{V_a}$ denote Lie derivatives along corresponding vector fields V_a. Needless to say that these equations cannot be in general solved explicitly, except in a few simple instances, e.g., planes, spheres and other highly symmetrical objects. In a general case, however, and as long as the $SO(3)$ group is taken to be relevant for describing disclinations, this approach seems to pose a problem.

Let us now turn to the second piece of the whole action, S_{gauge}, which describes a self-energy of disclinations. It acquires a standard form of the $SO(3)$ Yang-Mills action:

$$S_{gauge} = -\frac{s}{4} \int_{\Sigma_0} \sqrt{g} dx^1 dx^2 \langle \mathcal{F}^{ab}, \mathcal{F}_{ab} \rangle, \tag{5}$$

where s is a coupling strength, the form $\langle , \rangle$ stands for the $so(3)$ Killing trace and the $so(3)$-valued curvature tensor $\mathcal{F}_{ab} = \vec{F}_{ab} \cdot \vec{L}, \quad \vec{F}_{ab} = \partial_a \vec{W}_b - \partial_b \vec{W}_a + [\vec{W}_a, \vec{W}_b]$. The $so(3)$ generators L_i obey the following commutation rules

$$[L_i, L_j] = \epsilon_{ij}^k L_k, \quad i, j, k = 1, 2, 3,$$

where ϵ_{ij}^k stands for the fully antisymmetric tensor in R^3.

Finally, the Helfrich-Canham action that describes a self-energy of a fluctuating two-surface in R^3 looks like [9, 10]

$$S_{fl} = \frac{\kappa}{2} \int_{\Sigma_0} \sqrt{g} dx^1 dx^2 \, (trK)^2 + \frac{\kappa_G}{2} \int_{\Sigma_0} \sqrt{g} \, dx^1 dx^2 \, det \, g^{ab} K_{bc} \tag{6}$$

where κ is a bare bending rigidity and κ_G is a Gaussian rigidity,

$$K_{ab} = \vec{N} \cdot D_a D_b \vec{R} \tag{7}$$

is the curvature tensor, and $\vec{N}$ is the unit normal to the surface

$$\vec{N} = \frac{[\partial_1 \vec{R}, \partial_2 \vec{R}]}{|[\partial_1 \vec{R}, \partial_2 \vec{R}]|}.$$

The covariant derivative

$$D_a := \partial_a + \Gamma_a$$

includes the Levi-Civita connection Γ_a to be written down explicitly in the next section. Two scalar functions enter (6): $trK = g^{ab} K_{ab}$ called the mean (extrinsic) curvature, and $S = det \, g^{ab} K_{bc}$ referred to as the Gaussian (intrinsic) curvature. In view of the fact that the second piece on the r.h.s. of (6) is a topological invariant and depends only on the genus of the surface, it does not affect classical equations of motion. Incorporating disclinations amounts then to performing in the above formulas a substitution

$$\partial_a \to \nabla_a,$$

which results in the observation that the both pieces of action (6) will now contribute to the equations of motion.

3 Equations of motion

Equations of motion follow from the Hamilton principle of stationary action $\delta S = 0$ and read

$$\mathcal{D}_b\,\vec{\sigma}^b + \vec{J} = 0, \tag{8}$$

$$s\mathcal{D}_a\,\vec{F}^{ab} - \frac{1}{2}\,[\vec{R},\vec{\sigma}^b] + \frac{1}{2}\,\vec{I}^b = 0 \tag{9}$$

where a set of the stress vectors[4]

$$\vec{\sigma}^b = \frac{1}{2}(\nabla_a\vec{R})\,\rho^{ab}, \tag{10}$$

with

$$\rho^{ab} := \lambda\, g^{ab}\, tr E + 2\mu\, E^{ab} \tag{11}$$

has been introduced. The total covariant derivative

$$\mathcal{D}_a := \nabla_a + \Gamma_a$$

includes the Levi-Civita (torsion-free, metric compatible) connection

$$\Gamma^b_{ac} := (\Gamma_a)^b_c = \frac{1}{2}g^{bd}\left(\frac{\partial g_{dc}}{\partial x^a} + \frac{\partial g_{ad}}{\partial x^c} - \frac{\partial g_{ac}}{\partial x^d}\right) \tag{12}$$

to take care of a metric factor $\sqrt{g}$ in Eqs. (4-6) when a variation of the total action is calculated. It is noteworthy that Γ_a depends on $\vec{W}^{(0)}_a$ that enters in a definition of g_{ab}:

$$\begin{aligned} g_{ab} \;&=\; \nabla_a\vec{R}^{(0)}\cdot\nabla_b\vec{R}^{(0)} = \partial_a\vec{R}_{(0)}\cdot\partial_b\vec{R}_{(0)} + \partial_a\vec{R}^{(0)}[\vec{W}^{(0)}_b,\vec{R}^{(0)}] + \partial_b\vec{R}^{(0)}[\vec{W}^{(0)}_a,\vec{R}^{(0)}] \\ &+\; (\vec{W}^{(0)}_a\vec{W}^{(0)}_b)\vec{R}^2_{(0)} - (\vec{W}^{(0)}_a\vec{R}_{(0)})(\vec{W}^{(0)}_b\vec{R}_{(0)}). \end{aligned} \tag{13}$$

We have also abbreviated

$$\vec{J} = \frac{1}{\sqrt{g}}\frac{\delta S_{fl}}{\delta\vec{R}}, \quad \vec{I}^b = \frac{1}{\sqrt{g}}\frac{\delta S_{fl}}{\delta\vec{W}_b},$$

bearing in mind that explicit formulae are available in particular cases.

Obviously, for the elastic plane without defects (8) reduces to the well-known equilibrium equation $\partial_b\vec{\sigma}^b = 0$ while (9) is absent. In the general case, both the gauge fields and the affine connection enter (8) and (9) thus affecting stress fields. In view of (12)

$$\Gamma^b_{ba} = \frac{1}{2}\frac{\partial}{\partial x^a}\log g, \quad g = det||g_{ab}||$$

and one may consequently rewrite Eqs. (8,9) in the form

$$\frac{1}{\sqrt{g}}\partial_b\,(\sqrt{g}\vec{\sigma}^b) + [\vec{W}_b,\vec{\sigma}^b] + \vec{J} = 0 \tag{14}$$

and

$$\frac{1}{\sqrt{g}}\partial_a\,(\sqrt{g}F^{ab}) + \frac{1}{2s}[\vec{\sigma}^b,\vec{R}] + \frac{1}{2s}\,\vec{I}^b = 0, \tag{15}$$

respectively. As is seen, the basic self-consistent equations of the model are strongly nonlinear and it is difficult to get the general solution of the problem. It will be shown below, however, that for some physically interesting problems these equations become essentially simpler and can be solved explicitly.

[4] As was already mentioned, index a corresponds to the tangent space $T\Sigma_0$, whereas i is referred to the basis of the underlying space R^3. These are different manifolds and consequently $\vec{\sigma}^a = \{\sigma^a_i\}$ are viewed as a *set of vectors* in R^3. On the other hand, these spaces coincide in the EK approach and σ^a_i reduces to a stress *tensor*.

4 Applications

In this section we intend to consider three explicit realizations of the above theory. First, we show how the standard $2D$ EK theory emerges in the framework of our approach. Second, we consider an important in applications case of fluctuating elastic membranes, and, finally, a single disclination on arbitrary elastic surface is examined.

4.1 *Disclinations on elastic plane*

Let us start by recalling an explicit formulation of the $2D$ EK theory [19]. Consider a diffeomorphic map $\chi : D_0 \to D$, where $D_0 \in R^2$ appears as a strain free (which means $g_{ab} = \delta_{ab}$) domain and $D \in R^2$ stands for its image under deformation. Evidently, χ^a can be viewed upon as R^2-vector valued functions. Resulting action that describes dynamics of disclinations on a planar elastic body consists of two pieces S_{el} and S_{gauge} given by Eqs. (4) and (5), respectively, provided one puts

$$E_{ab} = \vec{B}_a \cdot \vec{B}_b - \delta_{ab}, \quad \vec{B}_a = \partial_a \vec{\chi} + [\vec{W}_a, \vec{\chi}],$$

where $\vec{W}_a = (0, 0, W_a)$. A complete set of the Euler-Lagrange equations can be easily derived and shown to possess exact vortex-like solutions [20].

On the other hand, basic notation of our theory in this case looks as follows. First we have the embeddings: $\vec{R}_{(0)} = (x, y, 0)$ and $\vec{R} = (R_1(x, y), R_2(x, y), 0)$, where x, y are Cortesian coordinates. Evidently, one has $\vec{W}_a = (0, 0, W_a)$, $\vec{F}^{ab} = (0, 0, F^{ab})$, where $F_{ab} = \partial_a W_b - \partial_b W_a$. The same representation holds for $\vec{W}_a^{(0)}$ which appears as a fixed gauge field associated with originally distributed disclinations.

Equations of motion (8) and (9) take the form ($\vec{J} = \vec{I}^a = 0$)

$$\frac{1}{\sqrt{g}} \partial_b \left(\sqrt{g} \vec{\sigma}^b \right) + [\vec{W}_b, \vec{\sigma}^b] = 0$$

$$\frac{1}{\sqrt{g}} \partial_a \left(\sqrt{g} F^{ab} \right) + \frac{1}{2s} [\vec{\sigma}^b, \vec{R}] = 0,$$

where $g = det||g_{ab}||$ and in view of Eq. (13)

$$g_{ab} = \delta_{ab} + \epsilon_{\alpha a} W_b^{(0)} R_{(0)}^{\alpha} + \epsilon_{\beta b} W_a^{(0)} R_{(0)}^{\beta} + (W_a^{(0)} W_b^{(0)}) \vec{R}_{(0)}^2, \quad \alpha, \beta = 1, 2.$$

Metric tensor g_{ab} is seen to deviate from its flat counterpart δ_{ab} at $W^{(0)} \neq 0$, whereby a nontrivial geometry is dynamically generated. It is at this point that our theory deviates from the standard EK one even in the trivial planar case.

At $W^{(0)} = 0$ the above equations reduce to those of the $2D$ EK theory [19]. In the linear approximation [21] no interaction between W_a and $W_a^{(0)}$ fields occurs, so that a total disclination flow $\oint \vec{\mathcal{W}} d\vec{l}$ that determines a source which gives rise to disclination-induced displacements, is found to be $\vec{\mathcal{W}} = \vec{W} - \vec{W}^{(0)}$.

4.2 *Disclinations on membranes*

Let us now turn to the gauge theory of defects for fluctuating membranes starting from (8,9), our aim being first that to recover a conventional Landau theory without defects. No disclinations

are assumed to be originally distributed as well, so that we put $\vec{W}_a^{(0)} = 0$. A flat membrane that fluctuates in the z-direction can be described by the embeddings:

$$\vec{R}^{(0)} = (x, y, 0), \quad \vec{R} = (x + u_x, y + u_y, f),$$

where $\vec{R} = \vec{R}^{(0)} + \vec{U}$ and $\vec{U} = (u_x, u_y, f)$ is a displacement of the $(x, y, 0)$ point under deformation. (In writing out the $\vec{U}$ components we stick to the Landau notation [7].) In accordance with the Landau theory $\vec{U}$ is assumed to be small in the transverse directions, along with the requirement that its z-component, $f(x, y)$, is to be viewed as a slowly varying function. Precise restrictions on components of $\vec{U}$ directly follow those in [7]. It is clear by obvious reasoning that within the adopted approximations only the z-component of $\vec{W}_a$ matters, so that we may put $\vec{W}_a = (0, 0, W_a)$. The strain tensor is then determined to be

$$\begin{aligned} E_{ab} &= \nabla_a \vec{R} \cdot \nabla_b \vec{R} - \delta_{ab} \\ &= \partial_a u_b + \partial_b u_a + \partial_a f \partial_b f + \epsilon_{\alpha a} W_b R^{\alpha}_{(0)} + \epsilon_{\alpha b} W_a R^{\alpha}_{(0)} + \mathcal{O}(u^2, u\partial f, W^2). \end{aligned} \tag{16}$$

with Greek indices $\alpha, \beta = 1, 2$ being used to specify coordinates of the plane orthogonal to the z-axes.

As for the curvature tensor K_{ab}, it is easily calculated to be

$$K_{ab} = \partial_a f \partial_b f + \mathcal{O}(u^2, u\partial f, W^2). \tag{17}$$

Within this accuracy one, consequently, obtains

$$trK = \Delta f.$$

To proceed with the equations of motion, we write down Eq. (8) in components referred to the z-direction and transverse plane, respectively:

$$\nabla_b \sigma_3^b + \kappa J_3 = 0$$

and

$$\nabla_b \sigma_\alpha^b + \kappa J_\alpha = 0.$$

Vector $\vec{J}$ is easily found to be $\vec{J} = (0, 0, \Delta^2 f)$, which yields for the above equations

$$\frac{1}{2}\partial_b(\partial_a f \, \rho^{ab}) + \kappa \, \Delta^2 f = 0, \tag{18}$$

and

$$\partial_b \rho^{b\beta} = 0, \tag{19}$$

respectively. Equation (11) now reads

$$\rho^{ab} = \lambda \, trE \, \delta^{ab} + 2\mu \, E^{ab},$$

with E^{ab} being given by (16).

Within the present approximation, $\vec{W}_a = (0, 0, W_a)$, and Eq. (9) is found to go over into

$$\partial_a F^{ab} = \frac{1}{4s} \epsilon_{\alpha\beta} \rho^{\beta b} R^{\alpha}_{(0)}. \tag{20}$$

In deriving this, we have also used that $I_3^a = 0$. Equations (18,19) coincide exactly at $W_a = 0$ with those in [7]. In our case, however, there appears an additional equation (20) which ensures the self-consistency of the model. As will be shown below, in the linear approximation this equation includes only gauge fields. In this case, one can use its solutions to determine the disclination-induced sources for the remaining equations.

Notice that the foregoing procedure which allowed to obtain $(18-20)$ is in agreement with the linearization scheme proposed within the EK model [14]. This scheme is based on a homogeneous scaling of the gauge group generators (see also details in [15, 18]). It can be shown that by applying this procedure to (8,9) one can get (18–20). It is important to note, however, that in accordance with this procedure one has to choose properly the relation between the model parameters (λ/s and μ/s). Depending on this choice one can describe elastic media with different properties. The classical elasticity theory which is of our interest here is recovered in the limit $\lambda/s \sim \epsilon$ and $\mu/s \sim \epsilon$ where ϵ is a scaling parameter. In this case, (20) reduces to

$$\partial_a F^{ab} = 0. \tag{21}$$

A singular vortex-like solution of (21) reads [20]

$$W_b = -\nu \epsilon_{bc} \partial_c \log r, \tag{22}$$

where ν is the Frank index.

One can easily see that (18) is rewritten as

$$(\partial_b \partial_a f)\sigma^{ab} = -\kappa^2 \Delta^2 f, \tag{23}$$

where (19) is taken into account. Let us consider (19). It resembles the usual equilibrium condition but the stress tensor includes the gauge fields. Within the linear approximation terms with W_b can be separated thus forming the source in the right-hand side of (19). In particular, for the planar case one can reproduce the known exact solution for a straight wedge disclination [21].

To compare our results with those in [1] let us differentiate Eq. (19), which yields

$$\partial_\beta \partial_b \rho^{b\beta} = 0.$$

After straightforward calculations one can rewrite this equation as

$$(\lambda/4\mu + 1/2)\,\Delta\, tr E = (\partial_x \partial_y f)^2 - \partial_x^2 f \partial_y^2 f + \epsilon_{\alpha b} \partial_\alpha W_b. \tag{24}$$

Notice that the last term in the right-hand side of (24) describes a source due to disclination fields. For solution (22) it takes the form

$$\epsilon_{\alpha b} \partial_\alpha W_b = \nu \Delta \log r = 2\pi\nu\delta(\vec{r}).$$

Introducing the Airy stress function $\chi(\vec{r})$ by $\sigma_{b\alpha} =: \epsilon_{bm}\epsilon_{\alpha n}\partial_m \partial_n \chi(\vec{r})$, one can finally rewrite equations (23) and (24) as

$$\begin{aligned} \kappa \Delta^2 f &= (\partial_y^2 \chi)(\partial_x^2 f) + (\partial_x^2 \chi)(\partial_y^2 f) - 2(\partial_x \partial_y \chi)(\partial_x \partial_y f), \\ K_0^{-1} \Delta^2 \chi &= (\partial_x \partial_y f)^2 - (\partial_x^2 f)(\partial_y^2 f) + 2\pi\nu\delta(\vec{r}), \end{aligned} \tag{25}$$

respectively. Here $K_0 = 4\mu(\lambda+\mu)/(\lambda+2\mu)$, and $trE = (1/(\lambda+\mu))\Delta\chi(\vec{r})$. As is seen, (25) are exactly the von Karman equations given in [1] for defects in hexatic membranes. It should be mentioned once more that the source term in (25) is not appeared *ad hoc* but generated by the gauge fields due to a disclination. Its exact form follows from the self-consistent solution of the basic model equations. An analysis of (25) shows [1] that isolated positive (five-fold) disclinations on free membranes buckle into a cone, while the negative (seven-fold) disclination leads to a saddle surface. The energy of a positive disclination was found to be less than that of a negative one. It is interesting that this asymmetry is absent in flat membranes and monolayers. Notice that the linear approximation used in this section allows to properly describe only the small out-of-plane fluctuations. Otherwise, the full set of equations (8,9) should be examined.

4.3 *Single disclination on arbitrary elastic surface*

In this subsection we demonstrate a non-trivial realization of the proposed model: a case of a single disclination on elastic surface. Let us consider a surface Σ_0 in a three-dimensional Euclidean space R^3. For any point $p \in \Sigma_0$ choose the z-axis to be normal to a tangent plane at p. Having in that way fixed the coordinate system in R^3, we consider in what follows Σ_0 as an embedding

$$(u,v) \to (R^x_{(0)}(u,v), R^y_{(0)}(u,v), R^z_{(0)}(u,v)),$$

with $x^1 = u, x^2 = v$ being the local coordinates on Σ_0. Under a local deformation concentrated at p, any nearby point undergoes a displacement

$$\vec{R}_{(0)}(u,v) \to \vec{R}(u,v) = (\vec{R}^{\perp}_{(0)}(u,v) + \vec{U}^{\perp}(u,v),\ R^z(u,v)).$$

As is above, the transverse displacement $\vec{U}^{\perp}$ is assumed to be small compared to the vertical one $U^z := R^z - R^z_{(0)}$, the latter being considered as a slowly varying function on a local chart that contains p. One may also put $\vec{W}_a = (0,0,W_a)$, which is in agreement with the assumptions made. The strain tensor then becomes

$$E_{ab} = \partial_a \vec{R}^{\perp}_{(0)} \partial_b \vec{U}^{\perp} + \epsilon_{\alpha\beta}\partial_a R^{\beta}_{(0)} R^{\alpha}_{(0)} W_b + (a \leftrightarrow b) + \partial_a R^z \partial_b R^z - \partial_a R^z_{(0)} \partial_b R^z_{(0)} + \mathcal{O}(U^2_{\perp}, W^2, U_{\perp} W). \tag{26}$$

In order to make a more close connection with the gauge theory of defects for fluctuating membranes of the preceding section, we find it appropriate to consider here a coordinate R^z as a dynamical variable rather than its displacement, U^z. In particular, the curvature tensor K_{ab} appears then as a direct covariant generalization of Eq. (17). Indeed, it is clear that

$$\vec{N}^{(0)}_p := \frac{[\partial_u \vec{R}_{(0)}, \partial_v \vec{R}_{(0)}]}{||[\partial_u \vec{R}_{(0)}, \partial_v \vec{R}_{(0)}]||}_{|p} = (0,0,1).$$

On the other hand, we have

$$K_{ab|_{p'}} = \vec{N}_{p'} \cdot D_a D_b \vec{R}_{|_{p'}} = (\vec{N}_p + \delta\vec{N}_p) \cdot D_a D_b \vec{R}_{|_{p'}}$$

$$= \vec{N}_p \cdot D_a D_b \vec{R}_{|_{p'}} + \mathcal{O}\left(||\delta\vec{N}_p||\right), \quad \delta\vec{N}_p := \vec{N}_{p'} - \vec{N}_p.$$

As $\vec{N}_p$ are sufficiently smooth and slowly varying functions of a reference point p, there exists a certain neighborhood of p, V_p, such that $||\delta\vec{N}_p|| \ll 1$ for any $p' \in V_p$.

In the linear approximation $\vec{N}_p$ can be replaced by its unperturbed value $\vec{N}_p^{(0)}$, so that for a local deformation of Σ_0 in the vicinity of p one obtains

$$K_{ab} = D_a D_b R^z$$

and consequently

$$tr\, K = g^{ab} D_a D_b R^z = D_a D^a R^z =: \Delta_{cov} R^z,$$

where the covariant Laplacian operator has been introduced.

Equation of motion (8) in the z and transverse components now reads

$$\frac{1}{2} D_b \left[(\partial_a R^z) \rho^{ab} \right] + \kappa\, \Delta^2_{cov} R^z = 0, \tag{27}$$

and

$$D_b \left[(\partial_a \vec{R}^{\perp}_{(0)}) \rho^{ab} \right] = 0, \tag{28}$$

respectively[5]. In the Monge gauge, $\vec{R}_{(0)} = (R^x_{(0)} = u =: x, R^y_{(0)} = v =: y, R^z_{(0)}(x,y))$, the last equation is written as

$$D_b \rho^{ab} = 0. \tag{29}$$

As is seen, (27) and (29) are nothing but a formally covariant representation of the Landau equations (18) and (19) with the nontrivial geometry of Σ_0 taken into account. These equations should be accompanied by the field equation (9) which in the linear approximation describes the $SO(2)$ gauge field in a curved background:

$$D_a F^{ab} = 0, \quad F^{ab} = \partial^a W^b - \partial^b W^a. \tag{30}$$

The following steps are just the same as in the previous subsection. A singular solution of (30) takes the form

$$W^b = -\nu \varepsilon^{bc} D_c G(x,y), \tag{31}$$

where $G(x,y)$ satisfies the equation

$$D_a D^a G(x,y) = 2\pi \delta^2(x,y)/\sqrt{g},$$

with $\varepsilon_{ab} = \sqrt{g} \epsilon_{ab}$ being the fully antisymmetric tensor on Σ_0. Note also that a conventional "flat" δ-function $\delta^2(x,y) := \delta(x)\delta(y)$ is to be multiplied by a metric factor $1/\sqrt{g}$ to make a scalar under a coordinate reparametrization.

Equation (27) now reads

$$\frac{1}{2} (D_b D_a R^z) \rho^{ab} = -\kappa^2 \Delta^2_{cov} R^z, \tag{32}$$

while (29) can be differentiated $D_a D_b \rho^{ab} = 0$ to yield

$$(\lambda/2\mu + 1)\, \Delta_{cov}\, tr E = (D_a D_b R^z)^2 - (\Delta_{cov} R^z)^2 - (R^z \leftrightarrow R^z_{(0)}) + 4\pi\nu \delta^2(x,y)/\sqrt{g}, \tag{33}$$

where the condition $\varepsilon^{ab} D_a W_b = \nu \Delta_{cov} G(x,y)$ has been taken into account. Obviously, (33) is the covariant analog of (24). It should be mentioned that due to the specific choice of the coordinate system related to a certain reference point, Eqs. (32) and (33) describe a *single* defect located at this point. In this regard, adding at least one more disclination makes the situation more difficult, so that turning to the basic equations of motion (8) and (9) seems necessary.

[5]Components of the three-vectors $\vec{R}(u,v)$ and $\vec{R}_{(0)}(u,v)$ appear as *scalar* functions with respect to a change of coordinates on Σ_0 and hence there is no need to replace the ordinary derivatives by the covariant ones in Eqs. (27, 28).

5 Conclusion

The model developed in this paper allows to describe disclinations on arbitrary elastic surfaces. It includes Riemannian surfaces that may change their geometry under deformations. In particular, within the proposed model one can study elastic properties of various materials containing disclinations: monolayers (flat surface), membranes (curved surface), fullerenes (spherical surface) as well as nanotubes (which can be considered as deformed spheres). For the flat surface, we have obtained the extended variant of the EK gauge theory that includes originally distributed disclinations. Within the linear scheme our model recovers the von Karman equations for membranes with a disclination-induced source being generated by gauge fields. For a single disclination on an arbitrary surface a covariant generalization of these equations is obtained.

In our opinion, the most intriguing application of this theory might be that for a disclination on a sphere. The obtained equations (27) and (29) are the most general ones which allow to study this problem properly. Notice, however, that this is a challenging task. Indeed, an analytical solution of the simpler system (18) and (19) has been obtained only in the limiting case $K_0 \to \infty$. Nevertheless, there are precise numerical solutions of (18) and (19) for arbitrary K_0 (see, e.g., [1] and the references therein). Obviously, any attempts to solve (27) and (29), either analytically or numerically would be of great interest.

Another important problem relevant for the physics of fullerenes and carbon nanotubes concerns the electronic properties of these materials. An attempt at extending the EK gauge theory of defects to include fermionic fields has been made in [22]. As has been shown, the self-consistent gauge model allows to describe physically interesting effects: the Aharonov-Bohm-like electron scattering due to disclinations [20], an electron localization near topological defects [18, 23] as well as a formation of the polaron-type states near dislocations [24]. We expect that incorporating fermions in the above-formulated theory may provide a new insight into disclination theory in a curved background in the presence of electrons and reveal some novel physical phenomena. Fortunately, our approach allows for a natural extension of the model to include fermions. Indeed, the model action (1) is constructed out of sections of the vector bundles over Σ_0. Considering on the other hand a spin bundle over Σ_0, that is a tangent bundle $T\Sigma_0 \to \Sigma_0$ with a structure group $Spin(2)$ amounts to incorporating fermions into the theory, with fermionic fields being local sections of the spin bundle. (We assume that Σ_0 admits a spin structure, which is the case, for instance, if Σ_0 appears as a Riemann surface with genus g.) Since $dim\Sigma_0 = 2$, the spin connection term drops out from an action, so that the Dirac operator on Σ_0 takes the form

$$D = i\gamma^a \partial_a + m, \quad \{\gamma^a, \gamma^b\} = g^{ab},$$

where $\gamma^a = e^a_\alpha \gamma^\alpha$, $g_{ab} = e^\alpha_a e^\beta_b \, \delta_{\alpha\beta}$ and the spin group $Spin(2)$ is generated by two Dirac matrices γ_α, $\alpha = 1, 2$ which can be taken as the Pauli matrices σ_1 and σ_2. An explicit form for the fermion action as well as a full set of ensuing equations of motion will be given elsewhere.

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